Quantum Signature of the Chaos–Order Transition in a Homogeneous SU(2) Yang–Mills–Higgs System [1]

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Abstract. We analyze a spatially homogeneous SU(2) Yang–Mills–Higgs system both in classical and quantum mechanics. By using the Toda criterion of the Gaussian curvature we find a classical chaos–order transition as a function of the Higgs vacuum, the Yang–Mills coupling constant and the energy of the system. Then, we study the nearest–neighbour spacing distribution of the energy levels, which shows a Wigner–Poisson transition by increasing the value of the Higgs field in the vacuum. This transition is a clear quantum signature of the classical chaos–order transition of the system.

1 Introduction

It is well known that the spatially uniform limit of scalar electrodynamics and Yang–Mills theory exhibits classical chaotic motion [1–8]. Usually the order–chaos transition in these systems has been studied numerically with Lyapunov exponents and sections of Poincarè. Less attention has been paid to analytical criteria.

[1]Presented to the VIII International Conference on *Symmetry Methods in Physics*, 27 July – 2 August 1997, Joint Institute for Nuclear Physics, Dubna (Russia).

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In this work we study analytically the suppression of classical chaos in the spatially homogenous SU(2) Yang–Mills–Higgs (YMH) system induced by the Higgs field. Then we analyze the energy fluctuation properties of the system, which give a clear quantum signature of the classical chaos–order transition of the system [9–13].

The SU(2) YMH system describes the interaction between a scalar Higgs field ϕ and three non–Abelian Yang–Mills fields A_μ^a, $a=1,2,3$. The Lagrangian density of the YMH system [14] is given by

$$L = \frac{1}{2}(D_\mu \phi)^+(D^\mu \phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} \; , \tag{1}$$

where

$$(D_\mu \phi) = \partial_\mu \phi - igA_\mu^b T^b \phi \; , \tag{2}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c \; , \tag{3}$$

with $T^b = \sigma^b/2$, $b=1,2,3$, generators of the SU(2) algebra, and where the potential of the scalar field (the Higgs field) is

$$V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4 \; . \tag{4}$$

We work in the (2+1)–dimensional Minkowski space ($\mu = 0,1,2$) and choose spatially homogeneous Yang–Mills and the Higgs fields

$$\partial_i A_\mu^a = \partial_i \phi = 0 \; , \quad i=1,2 \tag{5}$$

i.e. we consider the system in the region in which space fluctuations of fields are negligible compared to their time fluctuations.

In the gauge $A_0^a = 0$ and using the real triplet representation for the Higgs field we obtain

$$L = \dot{\vec{\phi}}^2 + \frac{1}{2}(\dot{\vec{A}}_1^2 + \dot{\vec{A}}_2^2) - g^2[\frac{1}{2}\vec{A}_1^2\vec{A}_2^2 - \frac{1}{2}(\vec{A}_1 \cdot \vec{A}_2)^2 + $$
$$+ (\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] - V(\vec{\phi}) \; , \tag{6}$$

where $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$, $\vec{A}_1 = (A_1^1, A_1^2, A_1^3)$ and $\vec{A}_2 = (A_2^1, A_2^2, A_2^3)$.

When $\mu^2 > 0$ the potential V has a minimum at $|\vec{\phi}| = 0$, but for $\mu^2 < 0$ the minimum is at

$$|\vec{\phi}_0| = \sqrt{\frac{-\mu^2}{4\lambda}} = v \; ,$$

which is the non zero Higgs vacuum. This vacuum is degenerate and after spontaneous symmetry breaking the physical vacuum can be chosen $\vec{\phi}_0 = (0,0,v)$. If $A_1^1 = q_1$, $A_2^2 = q_2$ and the other components of the Yang–Mills fields are zero, in the Higgs vacuum the Hamiltonian of the system reads

$$H = \frac{1}{2}(p_1^2 + p_2^2) + g^2 v^2 (q_1^2 + q_2^2) + \frac{1}{2} g^2 q_1^2 q_2^2 \ , \tag{7}$$

where $p_1 = \dot{q}_1$ and $p_2 = \dot{q}_2$. Here $w^2 = 2g^2v^2$ is the mass term of the Yang–Mills fields. This YMH Hamiltonian is a toy model for classical non–linear dynamics, with the attractive feature that the model emerges from particle physics. In the next sections we analyze first the classical chaos–order transition of the YMH system and then its connection to the quantal fluctuations of the energy levels.

2 Classical transition from chaos to order

In this paper we study the chaotic behaviour of this YMH system by using the Toda criterion of the Gaussian curvature of the potential energy [16]. The Toda criterion is based on a local estimation of the rate of separation of neighboring trajectories in the classical phase space of the model. To obtain the time evolution of a dynamical system with the Hamiltonian

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + V(q_1, q_2) \ , \tag{8}$$

the following Hamilton equations have to be solved:

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \ , \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} \ , \tag{9}$$

where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. The linearized equation of motion for the deviations are

$$\frac{d\delta\mathbf{p}}{dt} = I\delta\mathbf{p} \ , \qquad \frac{d\delta\mathbf{q}}{dt} = -S(t)\delta\mathbf{q} \ , \tag{10}$$

where I is the 2×2 identity matrix, and

$$S_{ij}(t) = \frac{\partial^2 V}{\partial q_i q_j}\Big|_{\mathbf{q}=\mathbf{q}(t)} \ , \tag{11}$$

with $\mathbf{q}(t)$ solution of the Hamilton equations. The stability of the dynamical system is then determined by the eigenvalues of the 4×4 stability matrix

$$\Gamma(\mathbf{q}(t)) = \begin{pmatrix} 0 & I \\ -S(t) & 0 \end{pmatrix} . \tag{12}$$

If at least one of the eigenvalues of the stability matrix Γ is real, then the separation of the trajectories grows exponentially and the motion is unstable. Imaginary eigenvalues correspond to stable motion.

To diagonalize the matrix Γ, we must first solve the equation of motion. The problem can be significantly simplified by assuming that the time dependence can be eliminated by replacement of the time–dependent point $\mathbf{q}(t)$ of configuration space by a time–independent coordinate $\mathbf{q}$, i.e. $\Gamma(\mathbf{q}(t)) = \Gamma(\mathbf{q})$. The eigenvalues then are

$$\lambda = \pm[-B \pm \sqrt{B^2 - 4C}]^{\frac{1}{2}} , \tag{13}$$

where

$$B = [\frac{\partial^2 V}{\partial q_1^2} + \frac{\partial^2 V}{\partial q_2^2}] , \tag{14}$$

$$C = [\frac{\partial^2 V}{\partial q_1^2}\frac{\partial^2 V}{\partial q_2^2} - (\frac{\partial^2 V}{\partial q_1 q_2})^2] . \tag{15}$$

Now, if $B > 0$ then with $C \geq 0$ the eigenvalues are purely imaginary and the motion is stable, while with $C < 0$ the pair of eigenvalues becomes real and this leads to exponential separation of neighboring trajectories, i.e. chaotic motion. The parameter c has the same sign as the Gaussian curvature K_G of the potential–energy surface:

$$K_G(q_1, q_2) = \frac{\frac{\partial^2 V}{\partial q_1^2}\frac{\partial^2 V}{\partial q_2^2} - (\frac{\partial^2 V}{\partial q_1 q_2})^2}{[1 + (\frac{\partial^2 V}{\partial q_1^2})^2 + (\frac{\partial^2 V}{\partial q_2^2})^2]^2} . \tag{16}$$

For our YMH system the potential energy is given by

$$V(q_1, q_2) = g^2 v^2 (q_1^2 + q_2^2) + \frac{1}{2} g^2 q_1^2 q_2^2 , \tag{17}$$

and the Γ matrix reads

$$\Gamma(q_1, q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2g^2v^2 - g^2 q_2^2 & -2g^2 q_1 q_2 & 0 & 0 \\ -2g^2 q_1 q_2 & -2g^2 v^2 - g^2 q_1^2 & 0 & 0 \end{pmatrix} \tag{18}$$

At low energy, the motion near the minimum of the potential, where the Gaussian curvature is positive, is periodic or quasi–periodic and is separated from the instability region by a line of zero curvature; if the energy is increased, the system will be, for some initial conditions, in a region of negative curvature, where the motion is chaotic. According to this scenario, the energy E_c of chaos–order transition is equal to the minimum value of the line of zero gaussian curvature $K(q_1, q_2)$ on the potential–energy surface. For our potential the gaussian curvature vanishes at the points that satisfy the equation

$$(2g^2v^2 + g^2q_2^2)(2g^2v^2 + g^2q_1^2) - 4g^4q_1^2q_2^2 = 0 \; . \tag{19}$$

It is easy to show that the minimal energy on the zero–curvature line is given by:

$$E_c = V_{min}(K_G = 0, \bar{q_1}) = 6g^2v^4 \; , \tag{20}$$

and by inverting this equation we obtain $v_c = (E/6g^2)^{1/4}$. We conclude that there is a order–chaos transition by increasing the energy E of the system and a chaos–order transition by increasing the value v of the Higgs field in the vacuum (see also [2]). Thus, there is only one transition regulated by the unique parameter $E/(g^2v^4)$.

It is important to point out that *in general* the curvature criterion guarantees only a *local instability* [16] and should therefore be combined with the Poincarè sections [17–18]. In the paper [19] we used a fourth–order Runge–Kutta method to compute numerically the classical trajectories and the Poincarè sections. The results confirm the analytical predictions of the curvature criterion: with $E = 10$ and $g = 1$ we get the critical value of the onset of chaos $v_c = (E/6g^2)^{1/4} \simeq 1.14$.

3 Quantum chaos and the Wigner distribution

The energy fluctuation properties of systems with underlying classical chaotic behaviour and time–reversal symmetry agree with the predictions of the Gaussian Orthogonal Ensemble (GOE) of random matrix theory, whereas quantum analogs of classically integrable systems display the characteristics of the Poisson statistics [9–12]. Some results in this direction for field theories have been obtained by Halasz and Verbaarschot: they studied the QCD

lattice spectra for staggered fermions and its connection to random matrix theory [13].

In quantum mechanics the generalized coordinates of the YMH system satisfy the usual commutation rules $[\hat{q}_k, \hat{p}_l] = i\delta_{kl}$, with $k, l = 1, 2$. Introducing the creation and destruction operators

$$\hat{a}_k = \sqrt{\frac{\omega}{2}}\hat{q}_k + i\sqrt{\frac{1}{2\omega}}\hat{p}_k \ , \quad \hat{a}_k^+ = \sqrt{\frac{\omega}{2}}\hat{q}_k - i\sqrt{\frac{1}{2\omega}}\hat{p}_k \ , \tag{21}$$

the quantum YMH Hamiltonian can be written [15]

$$\hat{H} = \hat{H}_0 + \frac{1}{2}g^2\hat{V} \ , \tag{22}$$

where

$$\hat{H}_0 = \omega(\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 + 1) \ , \tag{23}$$

$$\hat{V} = \frac{1}{4\omega^2}(\hat{a}_1 + \hat{a}_1^+)^2(\hat{a}_2 + \hat{a}_2^+)^2 \ , \tag{24}$$

with $\omega^2 = 2g^2v^2$ and $[\hat{a}_k, \hat{a}_l^+] = \delta_{kl}$, $k, l = 1, 2$.

The most used quantity to study the local fluctuations of the energy levels is the spectral statistics $P(s)$. $P(s)$ is the distribution of nearest–neighbour spacings $s_i = (\tilde{E}_{i+1} - \tilde{E}_i)$ of the unfolded levels $\tilde{E}_i$. It is obtained by accumulating the number of spacings that lie within the bin $(s, s + \Delta s)$ and then normalizing $P(s)$ to unit [9–12].

For quantum systems whose classical analogs are integrable, $P(s)$ is expected to follow the Poisson limit, i.e. $P(s) = \exp{(-s)}$. On the other hand, quantal analogs of chaotic systems exhibit the spectral properties of GOE with $P(s) = (\pi/2)s\exp{(-\frac{\pi}{4}s^2)}$, which is the so–called Wigner distribution [9–12]. The distribution $P(s)$ is the best spectral statistics to analyze shorter series of energy levels and the intermediate regions between order and chaos.

Seligman, Verbaarschot and Zirnbauer [20] analyzed a class of two–dimensional anharmonic oscillators with polynomial perturbation by using the Brody distribution [21]

$$P(s,\omega) = \alpha(\omega+1)s^{\omega}\exp{(-\alpha s^{\omega+1})} \ , \tag{25}$$

with

$$\alpha = (\Gamma[\frac{\omega+2}{\omega+1}])^{\omega+1} \ . \tag{26}$$

This distribution interpolates between the Poisson distribution ($\omega = 0$) of integrable systems and the Wigner distribution ($\omega = 1$) of chaotic ones, and thus the parameter ω can be used as a simple quantitative measure of the degree of chaoticity.

Higgs vacuum v	Brody parameter ω
1.00	0.99
1.05	0.47
1.10	0.34
1.15	0.12
1.20	0.01

The Table shows the calculated Brody parameter ω of the $P(s)$ distribution for different values of the Higgs vacuum v. There is Wigner–Poisson transition by increasing the value v of the Higgs field in the vacuum. Thus, by using the P(s) distribution and the Brody function, it is possible to give a quantitative measure of the degree of quantal chaoticity of the system. Our numerical calculations show clearly the quantum chaos–order transition and its connection to the classical one.

We compute the energy levels with a numerical diagonalization of the truncated matrix of the quantum YMH Hamiltonian in the basis of the harmonic oscillators [22]. If $|n_1 n_2>$ is the basis of the occupation numbers of the two harmonic oscillators, the matrix elements are

$$< n_1^{'} n_2^{'} | \hat{H}_0 | n_1 n_2 >= \omega (n_1 + n_2 + 1) \delta_{n_1^{'} n_1} \delta_{n_2^{'} n_2} \; , \tag{27}$$

and

$$< n_1^{'} n_2^{'} | \hat{V} | n_1 n_2 >= \frac{1}{4\omega^2} [\sqrt{n_1(n_1-1)} \delta_{n_1^{'} n_1 - 2} + \sqrt{(n_1+1)(n_1+2)} \delta_{n_1^{'} n_1 + 2} +$$

$$+ (2n_1+1) \delta_{n_1^{'} n_1}] \times [\sqrt{n_2(n_2-1)} \delta_{n_2^{'} n_2 - 2} + \sqrt{(n_2+1)(n_2+2)} \delta_{n_2^{'} n_2 + 2} + (2n_2+1) \delta_{n_2^{'} n_2}] \; . \tag{28}$$

The symmetry of the potential enables us to split the Hamiltonian matrix into 4 sub–matrices reducing the computer storage required. These sub–matrices are related to the parity of the two occupation numbers n_1 and n_2: even–even, odd–odd, even–odd, odd–even. The numerical energy levels depend on the dimension of the truncated matrix: we compute the numerical

levels in double precision increasing the matrix dimension until the first 100 levels converge within 8 digits (matrix dimension 1156×1156) [22–23].

We use the first 100 energy levels of the 4 sub–matrices to calculate the $P(s)$ distribution. In order to remove the secular variation of the level density as a function of the energy E, for each value of the coupling constant the corresponding spectrum is mapped, by a numerical procedure described in [24], into one which has a constant level density.

4 Conclusions

The chaotic behaviour of an homogenous YMH system has been studied both in classical and quantum mechanics. The Gaussian curvature criterion shows that the chaotic behaviour is regulated by the unique parameter $E/(g^2v^4)$. The YMH system has a order–chaos transition by increasing the energy E and a chaos–order transition by increasing the value v of the Higgs field in the vacuum.

The $P(s)$ distribution of the energy levels confirms with great accuracy the classical chaos–order transition of the YMH system. In particular, the Brody parameter ω shows a Wigner–Poisson transition for the $P(s)$ distribution in correspondence to the classical chaos–order transition.

Acknowledgments

The author is grateful to Dr. G.S Pogosyan and the organizing committee for their kind invitation to the Conference.

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